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ON EXISTENTIALLY CLOSED AND GENERIC NILPOTENT GROUPS†

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ABSTRACT

Let $n \geqq 2$ be an integer. We prove the following results that are known in case $n = 2$: The upper and the lower central series of an existentially closed nilpotent group of class n coincide. A finitely generic nilpotent group of class n is periodic and the center of a finitely generic torsion-free nilpotent group of class n is isomorphic to $\mathbf{Q}^+$, whereas infinitely generic nilpotent groups do not enjoy these properties. We determine the structure of the torsion subgroup of existentially closed nilpotent groups of class 2. Finally we give an algebraic proof that there exist 2^κ non-isomorphic existentially closed nilpotent groups of class n in cardinality $\kappa \geqq \aleph_0$.

0. Notation

Denote the class of nilpotent groups of class at most n by $\mathfrak{N}_n$ and the subclasses of torsion-free groups by $\mathfrak{N}_n^+$, the π-groups in $\mathfrak{N}_n$ by $\mathfrak{N}_n \cap L\mathscr{F}_\pi$ and the groups in $\mathfrak{N}_n$, whose torsion subgroups are π-groups, by $\mathfrak{N}_{n,\pi}$ where π is a set of primes. If $G \in \mathfrak{N}_n$, we denote the lower central series of G by $G = G_1 \geqq G_2 \geqq \cdots \geqq G_n \geqq G_{n+1} = 1$, where $G_{i+1} = [G_i, G]$, and the upper central series of G by $G = Z_n \geqq Z_{n-1} \geqq \cdots \geqq Z_1 \geqq Z_0 = 1$, where Z_{i+1}/Z_i is the center of G/Z_i. We write $[x, y]$ for the commutator $x^{-1}y^{-1}xy$ and adopt the usual left-normed notation $[x_0, \cdots, x_\mu]$ for the iterated commutator $[\cdots[[x_0, x_1], x_2], \cdots, x_\mu]$ and $[x, \mu y]$ for $[x, y, \cdots, y]$, where y appears μ times.

For the basic facts on algebraically closed (a.c.), existentially closed (e.c.), finitely generic (f.g.), and infinitely generic (i.g.) models the reader is referred to [3].

1. Introduction

Eklof and Sabbagh [2] opened a new period in the study of e.c. groups, when they showed that the class $\mathfrak{N}_1$ of Abelian groups has a model companion,

† Some results of this paper were contained in [6].

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whereas the class of all groups does not. Then Saracino proved that neither the class of soluble groups of derived length at most n ($n \geqq 2$) [13] nor the class of nilpotent groups of class at most n ($n \geqq 2$) [14] has a model companion. As to finitely generic groups, Saracino showed that G/G_2 in a f.g. metabelian group G is periodic and so are the centers Z_1 and Z_2/Z_1 in a f.g. nilpotent group of class at most n ($n \geqq 2$). In the case $n = 2$ this yields that a f.g. group in $\mathfrak{N}_2$ is periodic. We obtain this result for all $n \geqq 2$. As i.g. groups in $\mathfrak{N}_n$ are not periodic, the classes of f.g. and i.g. groups in $\mathfrak{N}_n$ ($n \geqq 2$) are disjoint, thus illustrating the non-existence of a model companion.

As to the torsion-free classes $\mathfrak{N}_n^+$ ($n \geqq 2$), Saracino proved the non-existence of a model companion in [14], and in [15] that the center of a f.g. group in $\mathfrak{N}_2^+$ is isomorphic to $\mathbf{Q}^+$. We obtain this last result for all $n \geqq 2$. Thus, in the torsion-free case as well, the classes of f.g. and i.g. groups in $\mathfrak{N}_n^+$ ($n \geqq 2$) are disjoint.

The class of countable e.c. groups in $\mathfrak{N}_2^+$ was studied independently by Baumslag and Levin [1] and Saracino [15]. Up to isomorphism there are only countably many countable e.c. groups $N_1 \leqq \cdots \leqq N_n \leqq \cdots \leqq N_\omega$ ($1 \leqq n \leqq \omega$) in $\mathfrak{N}_2^+$ and the center of N_n is isomorphic to the restricted direct product of n copies of $\mathbf{Q}^+$. Therefore, up to isomorphism there is only one f.g. group in $\mathfrak{N}_2^+$, namely N_1, and only one i.g. group, namely N_ω. We shall prove analogous results for $\mathfrak{N}_3^+$ in a subsequent paper (cf. [6], [10]).

As f.g. groups in $\mathfrak{N}_2$ are periodic, Saracino and Wood [16] considered periodic e.c. groups in $\mathfrak{N}_2$ and proved that there is up to isomorphism only one countable such group and thus only one f.g. group in $\mathfrak{N}_2$. As to the number of all countable e.c. groups, Hodges [4] proved by recursion theoretic methods that there are $2^{\aleph_0}$ pairwise non-elementarily equivalent e.c. groups in $\mathfrak{N}_{n,\pi}$ ($n \geqq 2$, π infinite). We illustrate this by sketching $2^{\aleph_0}$ countable groups in $\mathfrak{N}_{n,\pi}$ ($n \geqq 2$, π infinite) such that a countable group of this class can contain at most countably many of them, thus proving the existence of $2^{\aleph_0}$ pairwise non-isomorphic countable e.c. groups in $\mathfrak{N}_{n,\pi}$ ($n \geqq 2$, π infinite). We then consider the case of finite π and, for simplicity, $\pi = \{p\}$. We show that a divisible group $F \in \mathfrak{N}_{n,p}$ with $F_\mu = Z_{n+1-\mu}(F)$, $1 \leqq \mu \leqq n$ can be embedded in an e.c. group E such that F/F_3 is a distinguished subgroup of E/E_3. We achieve this by preventing all elements from $E \setminus FE_2$ from being p-divisible. This yields 2^κ e.c. groups in $\mathfrak{N}_{n,p}$ in each cardinality $\kappa \geqq \aleph_0$. All cases $\pi \neq \varnothing$ might be treated by the same method. $\pi = \varnothing$ is the torsion-free case $\mathfrak{N}_n^+$, where one should expect $\aleph_0$ countable e.c. groups as for $\mathfrak{N}_2^+$ and $\mathfrak{N}_3^+$.

Let us mention, in passing, that Saracino and Wood [16] showed by means of stability theory that there are 2^κ non-isomorphic f.g. groups of cardinality κ in $\mathfrak{N}_{2,\pi}$ ($n=2$, $\pi \neq \emptyset$, $\kappa > \aleph_0$). Their proof may be copied at least for the number of e.c. groups for all $n \geqq 2$ and arbitrary π.

We characterize the structure of the torsion subgroup of an e.c. group in $\mathfrak{N}_{2,\pi}$ as the direct product of a f.g. group $E_\pi \in \mathfrak{N}_{n,\pi}$ and a divisible abelian π-group T^∞. In the countable case E_π is unique whereas the p-ranks of T^∞ are arbitrary from $0, 1, \cdots, \aleph_0$. To obtain this result we show that a nilpotent group G can be embedded into a nilpotent group H of the same class whose maximal divisible subgroup consists of all elements of infinite height. We also study the adjunction of roots to e.c. nilpotent groups.

Saracino noted in [14] that every element in the center of an e.c. group in $\mathfrak{N}_n$ or $\mathfrak{N}_n^+$ ($n \geqq 2$) is an n-fold commutator. In the case $n=2$ this means that the center coincides with the commutator subgroup; this was the starting point for the classification of the countable e.c. groups in $\mathfrak{N}_2^+$ and the periodic ones in $\mathfrak{N}_2$ in [1], [15], [16]. We prove that the upper and the lower central series in an e.c. group in $\mathfrak{N}_n$ ($n \geqq 2$) coincide. This also holds for the subclasses $\mathfrak{N}_n^+$, $\mathfrak{N}_n \cap L\mathscr{F}_\pi$ and $\mathfrak{N}_{n,\pi}$. Moreover an element g in the $(\mu+1)$st term of the lower central series is a commutator of the form $g=[h, \mu x]$ in two variables h and x. The coincidence of the two natural central series indicates that the existential closure of a nilpotent group restricts the freedom of its characteristic and normal subgroup lattices. This was implicit in [1] and will be investigated for $\mathfrak{N}_3^+$ in the paper announced above (cf. [6], [10]). This homogeneity of an e.c. nilpotent group may be compared with the result of Neumann [11] that an e.c. group in the class of all groups is simple.

2. The upper and lower central series

Let $\mathscr{K}$ denote one of the classes $\mathfrak{N}_n$, $\mathfrak{N}_n^+$, $\mathfrak{N}_n \cap L\mathscr{F}_\pi$ or $\mathfrak{N}_{n,\pi}$ for a set π of primes and $n \geqq 2$. We exploit the existential closure of a nilpotent group in $\mathscr{K}$ via the existence of certain commutator relations. So let us quote the results I, II from [8] and III from [4].

I. If $M \leqq G \in \mathscr{K}$, $M \in \mathfrak{N}_m$ and $[M, G] \leqq G_{\mu+2}$ for m with $m(\mu+1) \leqq n$, then there exists a group $H \in \mathscr{K}$ such that $G \leqq H$ and that for all $g \in M$ there are $h, c \in H$ with $g = [h, \mu c]$. If G is periodic, h and c may be chosen periodic too ([8] Satz 4.2, p. 2185).

II. If $G \in \mathscr{K}$, $g_1, \cdots, g_k \in G$, $1 \neq xG_2 \in X \leqq G/G_2$, xG_2 independent from $\langle g_1, \cdots, g_k \rangle G_2$ and $g \in M \leqq Z$ such that $X \cong \mathbf{Q}^+ \cong M$, then there is a group

$H \in \mathcal{K}$ such that $G \leqq H$ and $g = [h, (n-1)x]$, $[h, g_i] = 1$ $(i = 1, \cdots, k)$ for some $h \in H$ ([8] Folgerung, p. 2183).

III. If $G \in \mathcal{K}$, xG_2 has infinite order in G/G_2 and $g \in G_n$, then there exists a group $H \in \mathcal{K}$ such that $G \leqq H$ and $g = [h_1, \cdots, h_{n-1}, x]$ for some $h_1, \cdots, h_{n-1} \in H$ ([4] Lemma 4).

We now prove

THEOREM 1. *The upper and the lower central series of an e.c. group G in $\mathcal{K}$ coincide. If $g \in G_{\mu+1}$, then $g = [h, \mu c]$ for some h, $c \in G$.*

PROOF. Let G be an e.c. group in $\mathcal{K}$. If $G = G_1 > G_2 > \cdots > G_{n+1} = 1$ denotes the lower central series and $1 = Z_0 < Z_1 \cdots < Z_n = G$ the upper central series of G respectively, we have to show $Z_{n-\mu} = G_{\mu+1}$ $(1 \leqq \mu \leqq n)$. Now '$\geqq$' holds in a nilpotent group of class n. So let us show '$\leqq$' by reverse induction on μ. The case $\mu = n$ is trivial, since $Z_0 = G_{n+1} = 1$. $\mu + 1 \rightarrow \mu$: If the assertion $Z_{n-\mu-1} \leqq G_{\mu+2}$ holds and $g \in Z_{n-\mu}$, then we have $\langle g \rangle \in \mathfrak{N}_1$ and $[\langle g \rangle, G] \leqq [Z_{n-\mu}, G] \leqq Z_{n-\mu-1} \leqq G_{\mu+2}$. The hypothesis of fact I being satisfied, we obtain a group $H \in \mathcal{K}$ such that $G \leqq H$ and $g = [h, \mu c]$ for some h, $c \in H$. As G is e.c. in $\mathcal{K}$ we have such elements in G. Thus $g \in G_{\mu+1}$ and $Z_{n-\mu} \leqq G_{\mu+1}$, as g was arbitrary in $Z_{n-\mu}$. Together with the induction we have proved the second assertion of the theorem. □

Observe that we only needed the closure of G with respect to one equation in two unknowns, so Theorem 1 also holds for a.c. groups in $\mathcal{K}$. If $G = 1$, then G is trivially a.c., but the assertions of the theorem are trivial too. If $G \neq 1$ is a.c. in $\mathcal{K}$, then $1 \neq Z_1 = G_n$ and $G \notin \mathfrak{N}_{n-1}$. As a consequence of Theorem 1 we have that the characteristic subgroups $G_{\mu+1} = Z_{n-\mu}$ $(\mu = 0, \cdots, n)$ of an e.c. group in $\mathcal{K}$ are defined by the $\exists$-formulas

$$\phi_{\mu+1}(x) \equiv \exists y \exists z [y, \mu z] = x$$

and also by the $\forall$-formulas

$$\psi_{n-\mu}(x) \equiv \forall y_1 \cdots \forall y_{n-\mu} [x, y_1, \cdots, y_{n-\mu}] = 1.$$

We are working in a first-order logic with the language $\{\cdot, ^{-1}, 1\}$ of group theory and use group-theoretic notions freely in our formulas. Observe that whereas $\psi_{n-\mu}$ defines $Z_{n-\mu}$ for all groups G, no first order formula can define $G_{\mu+1}$ for all $G \in \mathfrak{N}_n$ (cf. [5], p. 22).

3. Generic groups in $\mathfrak{N}_{n,\pi}$ for $n \geqq 2$

The classes $\mathfrak{N}_{n,\pi}$ can be axiomatized by Group $\cup\{\forall x_0 \cdots \forall x_n [x_0, \cdots, x_n] = 1, \forall x (x \neq 1 \rightarrow x^p \neq 1);\ p \notin \pi$ prime$\}$. So the notion of f.g. groups in $\mathfrak{N}_{n,\pi}$ is meaningful.

We first discuss the case $\pi \neq \emptyset$, where periodic elements are allowed.

Theorem 2. *A f.g. group in $\mathfrak{N}_{n,\pi}$ ($n \geqq 2$, $\pi \neq \emptyset$) is periodic. The finite and the infinite forcing companion are distinguished by an $\exists\forall\exists$-sentence.*

The classes of f.g. and i.g. groups are disjoint.

Proof. We define first-order formulas for $n \geqq 2$

$$\psi_n(v) \equiv \forall v_1 \cdots \forall v_n \exists w_1 \cdots \exists w_{n-1} [v_1, \cdots, v_n] = [w_1, \cdots, w_{n-1}, v]$$

and

$$\Psi_n \equiv \exists v \psi_n(v).$$

(1) *If Ψ_n holds in an e.c. group G in $\mathfrak{N}_{n,\pi}$, then G contains an element of infinite order modulo G_2.* For if Ψ_n holds in G, then there is some $x \in G$ such that $\psi_n(x)$ holds in G. Now let $y_1, \cdots, y_n \in G$ and $h_1, \cdots, h_{n-1} \in G$ satisfy $[y_1, \cdots, y_n] = [h_1, \cdots, h_{n-1}, x]$. If $o([y_1, \cdots, y_n]) > m$, it follows from

$$1 \neq [y_1, \cdots, y_n]^m = [h_1, \cdots, h_{n-1}, x]^m = [h_1, \cdots, h_{n-1}, x^m]$$

that $o(xG_2) \nmid m$, as $[G_{n-1}, G_2] = 1$. Because m can be chosen arbitrarily large in the e.c. group G, we obtain $o(xG_2) = \infty$.

(2) *If G is e.c. in $\mathfrak{N}_{n,\pi}$ and contains an element of infinite order modulo G_2, then Ψ_n holds in G.* For it follows from fact III that G satisfies $\psi_n(x)$ for every x of infinite order modulo G_2.

(3) *Ψ_n does not hold in any f.g. group in $\mathfrak{N}_{n,\pi}$.* Assume to the contrary that the sentence Ψ_n holds in the f.g. group G. Then it is forced by a forcing-condition p_0 of the diagram of G, i.e. a finite part of the group table, where we do not distinguish between group elements and forcing constants. Then p_0 forces the formula $\psi_n(x)$ for some $x \in G$. This means that for all conditions $p_1 \supset p_0$ and all constants $y_1, \cdots, y_n \in G$ there exists a condition $p_2 \supset p_1$ which forces the formula

$$\exists w_1 \cdots \exists w_{n-1} [y_1, \cdots, y_n] = [w_1, \cdots, w_{n-1}, x]$$

and this holds if and only if p_2 forces $[y_1, \cdots, y_n] = [h_1, \cdots, h_{n-1}, x]$ for some $h_1, \cdots, h_{n-1} \in G$.

Let us choose $p_1 = p_0 \cup \{x^m = 1\}$ for some π-number $m \geqq 1$. Such a number m

exists as a finitely generated group $H \in \mathfrak{N}_{n,\pi}$, which realizes p_0, is a residually finite π-group (cf. [5], 17.2.2), and therefore has a finite factor $\bar{H}$ in $\mathfrak{N}_{n,\pi}$ still realizing p_0. Here we have used $\pi \neq \emptyset$. If we let m be the order of $\bar{x}$ in $\bar{H}$, then p_1 is a condition, because an isomorphic copy of the finite group $\bar{H} \in \mathfrak{N}_{n,\pi}$ is contained in the e.c. group G. If we now choose $y_1, \cdots, y_n \in G$ such that $[y_1, \cdots, y_n]^m \neq 1$, then we come to a contradiction with $x^m = 1$ as in (1).

(4) By (2) and (3) the commutator factor group G/G_2 of a f.g. group $G \in \mathfrak{N}_{n,\pi}$ is periodic. Inductively, all factor groups $G_\mu / G_{\mu+1}$ are periodic and thus G is periodic.

(5) *An i.g. group in $\mathfrak{N}_{n,\pi}$ satisfies Ψ_n.* As $\mathfrak{N}_{n,\pi}$ has the joint embedding property, we may take an i.g. group H in $\mathfrak{N}_{n,\pi}$ containing G and an element of infinite order modulo $Z_{n-1}(H) = H_2$. Thus Ψ_n holds in H by (2) and therefore in G, because the i.g. group G is an elementary substructure of the i.g. group H (cf. [3], I.3.4).

(6) By (5) the $\exists\forall\exists$-sentence Ψ_n is an element of the infinite forcing companion of $\mathfrak{N}_{n,\pi}$, whereas by (3) and, as the language is countable, $\neg \Psi_n$ is contained in the finite forcing companion (cf. [3], I.5.19). Because f.g. and i.g. groups are models of the finite and infinite forcing companions respectively, these two classes are disjoint in $\mathfrak{N}_{n,\pi}$. □

We now consider the torsion-free case. Here, by a theorem of Mal'cev, we know that a torsion-free nilpotent group can be embedded in a divisible one of the same class (cf. [5], 17.3.2). So an e.c. group in $\mathfrak{N}_n^+$ is divisible and, as the factors Z_{i+1}/Z_i of the upper central series are torsion-free, we have that $G/G_2 = G/Z_{n-1}$ is a divisible torsion-free abelian group and an element $1 \neq xG_2$ lies in a direct factor of G/G_2 isomorphic to $\mathbf{Q}^+$.

THEOREM 3. *In a f.g. group in $\mathfrak{N}_n^+$, $n \geqq 2$ the center is isomorphic to $\mathbf{Q}^+$, whereas in an i.g. group it has infinite dimension (as a $\mathbf{Q}$-space). The finite and the infinite forcing companions are distinguished by an $\exists\forall\exists$-sentence and the classes of f.g. and i.g. groups in $\mathfrak{N}_n^+$ are disjoint.*

Note that Saracino [15] proved that a countable e.c. group in $\mathfrak{N}_2^+$ is f.g. if and only if its center has dimension 1. This follows from the classification of the countable e.c. groups in $\mathfrak{N}_2^+$. We shall prove it for $\mathfrak{N}_3^+$ in the announced paper (cf. [10]).

PROOF. For $d \geqq 1$ we define first-order formulas

$$\phi_d(u, v_1, \cdots, v_d, w, v) \equiv [w,(n-1)u] = [v,(n-1)u] \wedge \forall u_0 \left(\bigwedge_{i=1}^{d} [u_0, v_i] = 1 \rightarrow [u_0, v] = 1 \right),$$

$$\Phi_d \equiv \exists u \exists v_1 \cdots \exists v_d \forall w \exists v ([v_1,(n-1)u] \neq 1 \wedge \phi_d(u, v_1, \cdots, v_d, w, v))$$

and

$$\Phi_1' \equiv \forall u \forall v_1 \forall w \exists v ([v_1,(n-1)u] \neq 1 \rightarrow \phi_1(u, v_1, w, v)).$$

(1) *If Φ_d holds in an e.c. group G in $\mathfrak{N}_n^+$, then* $\dim Z \leqq d$. If Φ_d holds in the group G then there are elements $x, h_1, \cdots, h_d \in G$ such that the formula

$$\forall w \exists v \left([w,(n-1)x] = [v,(n-1)x] \wedge \forall u_0 \left(\bigwedge_{i=1}^{d} [u_0, h_i] = 1 \rightarrow [u_0, v] = 1 \right) \right)$$

holds in G. This means that for any $g \in G$ there exists some $h \in G$ which satisfies $[g,(n-1)x] = [h,(n-1)x]$ and which is linearly dependent of $\langle h_1, \cdots, h_d \rangle$ modulo G_2 by fact II. Now $[g',(n-1)x] = 1$ for $g' \in G_2$ and $[h^r,(n-1)x] = [h,(n-1)x]^r$ for $r \in \mathbf{Q}$ in the group $G \in \mathfrak{N}_n^+$. Thus the center $Z = G_n$ of the e.c. group G is spanned by $[h_i,(n-1)x]$ $(i = 1, \cdots, d)$ as a $\mathbf{Q}$-space and so has dimension at most d. Similarly, one sees that an e.c. group in $\mathfrak{N}_n^+$ satisfying Φ_1' has center isomorphic to $\mathbf{Q}^+$.

(2) *If the center of an e.c. group G in $\mathfrak{N}_n^+$ has dimension at most d, then Φ_d holds in G.* Take some $x \notin G_2$. By fact II we can find $h_1, \cdots, h_d \in G$ such that the elements $z_i = [h_i,(n-1)x]$ $(i = 1, \cdots, d)$ span the center Z and furthermore $[h_1,(n-1)x] \neq 1$. Now if g is another element of G, then $[g,(n-1)x] \in Z$ and we have for suitable $r_1, \cdots, r_d \in \mathbf{Q}$

$$\begin{aligned}[g,(n-1)x] &= \prod_{i=1}^{d} z_i^{r_i} \\ &= \prod_{i=1}^{d} [h_i,(n-1)x]^{r_i} \\ &= \left[\prod_{i=1}^{d} h_i^{r_i}, (n-1)x \right] = [h,(n-1)x]\end{aligned}$$

where we have set $h = \prod_{i=1}^{d} h_i^{r_i}$. Since centralizers in torsion-free groups are isolated (cf. [5], 16.2.9), the element $h \in G$ satisfies the formula

$$\bigwedge_{i=1}^{d} [y, h_i] = 1 \rightarrow [y, h] = 1 \qquad \text{for all } y \in G.$$

Thus G satisfies Φ_d. In case $d = 1$ the center is spanned by any n-fold commutator different from 1, so we have that Φ'_1 holds in this case, too.

(3) *The theory of* $\mathfrak{N}_n$ *forces the sentence* Φ'_1. Let a forcing condition p_0 and constants x, h_1, g be given. We have to show that there exists a condition $p_1 \supset p_0$ and a constant h such that p_1 forces the formula

$$\gamma \equiv [h_1, (n-1)x] \neq 1 \rightarrow \phi_1(x, h_1, g, h).$$

Take a group $G \in \mathfrak{N}_n^+$, that realizes p_0. If the equation $[h_1, (n-1)x] = 1$ holds in G, then we may set

$$p_1 = p_0 \cup \{[h_1, (n-1)x] = 1\},$$

and p_1 forces γ for any constant h.

Now assume that $[h_1, (n-1)x] \neq 1$ holds in G. Let us show that we can also assume that G has center isomorphic to $\mathbf{Q}^+$. First note that G may be supposed to be the divisible hull of a finitely generated group, since p_0 is finite. Refining the upper central series, the factors of which are finitely generated torsion-free abelian groups, we obtain a central series $G = N_1 > \cdots > N_{k+1} = 1$ of G such that all factors are isomorphic to $\mathbf{Q}^+$. Now if $Z \geqq N_{k-1}$, let K be the kernel of a homomorphism from N_{k-1} onto N_k, such that the finitely many inequalities from $p_0 \cup \{[h_1, (n-1)x] \neq 1\}$ that have to hold in N_{k-1} still hold in N_{k-1}/K. Since equations are preserved under homomorphisms, the factor group G/K also satisfies the condition $p_0 \cup \{[h_1, (n-1)x] \neq 1\}$. But as $K \cong \mathbf{Q}^+$, G/K has a central series with factors isomorphic to $\mathbf{Q}^+$ of length strictly less than k. Thus after a finite number of quotients we shall satisfy $p_0 \cup \{[h_1, (n-1)x] \neq 1\}$ in a group $G \in \mathfrak{N}_n^+$ with center isomorphic to $\mathbf{Q}^+$.

Now as $[h_1, (n-1)x] \neq 1$, we can assume that $[g, (n-1)x]^j = [h_1, (n-1)x]^k$ for some $j, k \in \mathbf{Z}$, $j \neq 0$ and thus $[g, (n-1)x] = [h, (n-1)x]$ for $h = h_1^{k/j}$. Therefore

$$p_1 = p_0 \cup \{[h_1, (n-1)x] \neq 1,\ [g, (n-1)x] = [h, (n-1)x],\ h^j = h_1^k\}$$

is a condition. Since the inequality $[h_1, (n-1)x] \neq 1$ is contained in p_1, the condition p_1 forces the formula γ, if it forces $\phi_1(x, h_1, g, h)$. Now the first part $[g, (n-1)x] = [h, (n-1)x]$ of the conjunction $\phi_1(x, h_1, g, h)$ is contained in p_1. So let us show that p_1 also forces the second one $\forall u_0([u_0, h_1] = 1 \rightarrow [u_0, h] = 1)$. For this let $p_2 \supset p_1$ be another condition and y a constant, and take a group $G \in \mathfrak{N}_n^+$, which realizes p_2.

If the inequality $[y, h_1] \neq 1$ holds in G, then set $p_3 = p_2 \cup \{[y, h_1] \neq 1\}$. If $[y, h_1] = 1$ holds in G, then since $h^j = h_1^k \in p_2$ also $[y, h] = 1$ holds; set $p_3 =$

$p_2 \cup \{[y, h_1] = 1, [y, h] = 1\}$ in this case. In both cases p_3 is a condition that forces the formula $[y, h_1] = 1 \rightarrow [y, h] = 1$.

With both parts p_1 now forces $\phi_1(x, h_1, g, h)$.

(4) *An i.g. group G in $\mathfrak{N}_n^+$ has center of infinite dimension.* This follows from the joint embedding property and so (1) implies that an i.g. group satisfies $\neg \Phi_d$ for all $d \geqq 1$.

(5) *By* (3) *and* (4) *the* $\forall\exists\forall$*-sentence* Φ_1' *belongs to the finite and* $\neg \Phi_1'$ *to the infinite forcing companion of* $\mathfrak{N}_n^+$. Therefore the classes of f.g. and i.g. groups in $\mathfrak{N}_n^+$ are disjoint. □

4. The torsion subgroup of e.c. groups in $\mathfrak{N}_{n,\pi}$

The torsion subgroup of a group $G \in \mathfrak{N}_{n,\pi}$ is the direct product of its maximal p-subgroups for $p \in \pi$. In the case $\pi = \{p\}$ we shall write $\mathfrak{N}_{n,p}$. Recall that an element g is called of infinite height in G, if there is an m-th root for g in G for all $m \geqq 1$.

We start with some lemmas that may be interesting in their own right.

LEMMA 4.1. *$G \in \mathfrak{N}_{n,\pi}$ may be embedded into some $H \in \mathfrak{N}_{n,\pi}$ such that every element of infinite height in H lies in a divisible subgroup.*

PROOF. First note that an element of infinite height in G has infinite height in any group containing G. Thus by a classical tower argument, it suffices to construct a H such that one fixed element x of infinite height in G lies in a divisible subgroup of H. This is clear from the compactness theorem. But let us also give a group-theoretic proof.

Take the n-th nilpotent product N of G and infinite cyclic groups $\langle c_i \rangle$ ($i \in \mathbf{N}$). Then $N \in \mathfrak{N}_{n,\pi}$. Consider the normal subgroup K generated by $\{xc_1^{-1}, c_i c_{i+1}^{-i+1};\ i \in \mathbf{N}\}$. We want to show that $G \cap K = 1$. An element $h \in G \cap K$ is a finite product of conjugates of the generators. Call k the highest index of a c_i in the product for h. Now by hypothesis x has a $(k!)$-th root in G, g say. The identity on G and $c_i \mapsto g^{(k!/i!)}$ $(1 \leqq i \leqq k)$, $c_i \mapsto 1$ $(i > k)$ induce a homomorphism from N onto G. By choice of the images of the c_i $(1 \leqq i \leqq k)$, h lies in the kernel of this homomorphism. On the other hand $h \in G$ is mapped identically, hence $h = 1$. Thus $G \cap K = 1$, and G is embedded into N/K. By factoring out the maximal π'-subgroup of N/K we obtain a factor group $\bar{N} \in \mathfrak{N}_{n,\pi}$ such that still $G \leqq \bar{N}$ and x lies in the divisible subgroup $\langle x, \bar{c}_i;\ i \in \mathbf{N} \rangle$ of $\bar{N}$. □

The following is a generalization of the well known facts that quasi-cyclic

p-subgroups of a nilpotent group are contained in the center of the torsion subgroup and that the torsion subgroup of a divisible nilpotent group is central.

The results concerning the centralizer of the torsion subgroup were proved directly by Warfield [17, theorem 6.13].

COROLLARY 4.2. *$G \in \mathfrak{N}_{n,\pi}$ may be embedded in some $H \in \mathfrak{N}_{n,\pi}$ such that the periodic elements of infinite height in H form a divisible subgroup of the center of the torsion subgroup. Thus, periodic elements of infinite height in G are central in the torsion subgroup.*

More generally, H may be chosen such that the elements of infinite height form its maximal divisible subgroup; this subgroup is contained in the centralizer of the torsion subgroup.

PROOF. By the preceding remark, the divisible subgroups for the periodic elements of infinite height in some group H as in the lemma generate a divisible subgroup of the center of the torsion subgroup of H. This implies the first assertions.

For the last assertion it suffices to show: if $G \in \mathfrak{N}_n$, T its torsion subgroup and X a subgroup isomorphic to $\mathbf{Q}^+$, then $X \leqq C_G(T)$. Proceeding inductively, we may assume that $X \leqq C_G(T \cap Z_i)$. Take any $y \in T \cap Z_{i+1}$. Then $y^q = 1$ for some $q \neq 0$ and $[x, y] \in T \cap Z_i$ for all $x \in X$. Hence, if $x'^m = x$ in X, it follows that $[x, y] = [x'^m, y] = [x', y]^m$. So $[x, y] \in Z(T)$ for all $x \in X$, since $[x, y]$ is a periodic element of infinite height. Finally, if $x'^q = x$ in X, we conclude that $[x, y] = [x', y]^q = [x', y^q] = 1$. Therefore $X \leqq C_G(T \cap Z_{i+1})$. □

Mal'cev's theorem that a torsion-free nilpotent group may be embedded into a divisible nilpotent group of the same class cannot be generalized to nilpotent groups with non-trivial torsion subgroup. So we need some information on the existence of roots in nilpotent groups. The following lemma gives such a condition for elements in the second center Z_2 of an e.c. group G in $\mathfrak{N}_{n,\pi}$. We denote the centralizer of $g_1, \cdots, g_r$ in a group H by $C_H(g_1, \cdots, g_r)$ and drop the subscript for $H = G$.

LEMMA 4.3. *Let G be e.c. in $\mathfrak{N}_{n,\pi}$, $g_1, \cdots, g_r \in G$ and $g \in Z_2 \cap C(g_1, \cdots, g_r)$. There exists an element $h \in Z_2 \cap C(g_1, \cdots, g_r)$ with $h^p = g$, if and only if $x \in C(g)$ for all $x \in G$ with $x^p \in C(Z_2 \cap C(g, g_1, \cdots, g_r))$.*

PROOF. We set $g_0 = g$.

Necessity: Let $h^p = g$ for some $h \in Z_2 \cap C(g_1, \cdots, g_r)$. If $x \in G$ with $x^p \in C(Z_2 \cap C(g_0, \cdots, g_r))$ then $1 = [h, x^p] = [h, x]^p = [h^p, x] = [g, x]$.

Sufficiency: By the proof of Theorem 1 it suffices to embed G into a group $H \in \mathfrak{N}_{n,\pi}$ such that $h^p = g$ for some $h \in Z_2(H) \cap C_H(g_1, \cdots, g_r)$. Let $H = G(2)\langle c \rangle$ be the second nilpotent product of G with an infinite cyclic group $\langle c \rangle$. Let N be the normal subgroup of H that is generated by $c^p g$ and $[c, g_1], \cdots, [c, g_r]$. Then $H \in \mathfrak{N}_{n,\pi}$, $c \in Z_2(H)$ and $N \leqq Z_2(H)$. We now prove $G \cap N = 1$. An element $h \in G \cap N$ can be written in the form $h = \prod_{i=1}^k ((c^p g)^{\varepsilon_i})^{h_i} \cdot [c, g_1]^{n_1} \cdots [c, g_r]^{n_r}$ with $\varepsilon_i = \pm 1$, $h_i \in H$ and $n_i \in \mathbf{Z}$. Let us calculate.

$$h = \prod_{i=1}^{k} (c^p g)^{\varepsilon_i} [(c^p g)^{\varepsilon_i}, h_i] \cdot [c, g_1^{n_1}] \cdots [c, g_r^{n_r}]$$

$$= (c^p g)^{\varepsilon'} \cdot [c^p g, h'] \cdot [c, g'],$$

where $\varepsilon' = \sum_{i=1}^k \varepsilon_i \in \mathbf{Z}$, $h' = \prod_{i=1}^k h_i^{\varepsilon_i}$ and $g' = \prod_{i=1}^r g_i^{n_i} \in \langle g_1, \cdots, g_r \rangle$. If we write $h' = h_G \cdot c^l \cdot h''$ with $h_G \in G$, $l \in \mathbf{Z}$ and $h'' \in H_2$ then

$$h = c^{p\varepsilon'} g^{\varepsilon'} [g, c^p]^{\binom{\varepsilon'}{2}} \cdot [c^p, h_G][g, h_G][g, c^l] \cdot [c, g'].$$

We now use the universal property of H: If $F \in \mathfrak{N}_n$ then any homomorphisms $f_G : G \to F$ and $f_c : \langle c \rangle \mapsto Z_2(F)$ can be lifted simultaneously to a homomorphism $f : H \to F$. This follows from the fact that H is a factor group of the free product $G * \langle c \rangle$ and from the condition on f_c. We shall only use the special case $F = G$, $f_G = \mathrm{id}_G$ and f_c a homomorphism which is given by $c \mapsto z \in Z_2$. Since we assumed $h \in G \cap N$ any choice of $z \in Z_2$ will leave h fixed. From $c \mapsto 1$ we deduce

$$h = g^{\varepsilon'} \cdot [g, h_G] \quad \text{and} \quad c^{p\varepsilon'} [g, c^p]^{\binom{\varepsilon'}{2}} \cdot [c^p, h_G][g, c^l] \cdot [c, g'] = 1.$$

If we choose $c \mapsto z \in Z$, we obtain $z^{p\varepsilon'} = 1$ and $o(z) \mid p\varepsilon'$. Since z was arbitrary in Z we conclude $\varepsilon' = 0$ and hence

$$h = [g, h_G] \quad \text{and} \quad [c^p, h_G][g, c^l] \cdot [c, g'] = 1.$$

The assertion $h = 1$ will follow from the hypothesis on g, if we can show that $h_G^p \in C(Z_2 \cap C(g_0, \cdots, g_r))$. For this take $c \mapsto z \in Z_2 \cap C(g_0, \cdots, g_r)$. Then $[g, z^l] = 1 = [z, g']$ and hence $1 = [z^p, h_G] = [z, h_G]^p = [z, h_G^p]$. Now G is embedded into H/N, since $G \cap N = 1$. If $\bar{H}$ denotes the factor group of H/N by its π'-torsion subgroup, we obtain $G \leqq \bar{H} \in \mathfrak{N}_{n,\pi}$, $\bar{h}^{-p} = g$ and $\bar{h} \in Z_2(\bar{H}) \cap C(g_0, \cdots, g_r)$. □

We need two results on centralizers in the proof of our next proposition. We denote the π-isolator of a subgroup H in G by $I_\pi(H)$; this is the set of all $g \in G$ such that some π-power of g lies in H.

Recall that centralizers in a π'-torsion-free nilpotent group G are π'-isolated, i.e. $I_{\pi'}(C(H)) = C(H)$. This holds, for example, if $G \in \mathfrak{N}_{n,\pi}$.

LEMMA 4.4. *Let G be e.c. in $\mathfrak{N}_{n,\pi}$.*

(1) $C(Z_2) = G_2$.

(2) *If $n = 2$ and $g \in G$, then* $C^2(g) := C(C(g)) = I_{\pi'}(\langle g \rangle)Z$.

PROOF. Let us quote a variant of Lemma 3 from Hodges [4]: If G is an e.c. group in $\mathfrak{N}_{n,\pi}$ and $a, b \in G$, then there exist $g_1, \cdots, g_{n-1} \in G$ with $[g_1, \cdots, g_{n-1}, a] = 1 \neq [g_1, \cdots, g_{n-1}, b]$ if and only if $b \notin I_{\pi'}(\langle a \rangle)G_2$. Setting $a = 1$ we obtain $b \notin C(G_{n-1}) = C(Z_2)$, if $b \notin G_2 = I_{\pi'}(1)G_2$. Hence $C(Z_2) \leqq G_2$, and the reverse inclusion holds in every $G \in \mathfrak{N}_n$. For the second assertion it again suffices to show '$\leqq$', since $g \in C^2(g)$. So assume $b \notin I_{\pi'}(\langle g \rangle)Z$. Setting $a = g$ we obtain an element $x \in G$ such that $[x, g] = 1 \neq [x, b]$. Thus $x \in C(g)$ and $b \notin C(x) \geqq C^2(g)$. □

Observe that the first assertion says that the centralizer of Z_2 is as small as possible. It would be interesting to know whether the analogous assertions hold for the higher centers. This would imply that Z_μ in an e.c. group is defined not only by the formula $\forall y_1 \cdots \forall y_\mu [x, y_1, \cdots, y_\mu] = 1$, but also by the formula $\forall y_1 \cdots \forall y_\mu [y_1, \cdots, y_\mu, x] = 1$, since then also

$$C(G_\mu) = C(Z_{n+1-\mu}) = G_{n+1-\mu} = Z_\mu.$$

The second assertion tells us that the centralizer of a single element g is rather large, as the second centralizer $C^2(g)$ of the element is as small as possible. Another aspect of this is that elements are determined by their centralizers as precisely as possible:

COROLLARY 4.5. *Let G be e.c. in $\mathfrak{N}_2$ and $g, h \in G$. Then $C(g) = C(h)$ if and only if $\langle g \rangle Z = \langle h \rangle Z$.*

PROOF. The condition is clearly sufficient. It is necessary by the lemma, since in case $\pi' = \varnothing$ we have $I_{\pi'}(\langle g \rangle) = \langle g \rangle$. □

We are now in a position to reformulate our condition for the existence of p-th roots of elements in the second center of an e.c. nilpotent group. Although being a rather restricted case for $n > 2$, this will turn out quite useful. In the case $n = 2$ it may be looked at as a generalization of the main step in the proof of Mal'cev's theorem, namely the adjunction of a p-th root to an element in a torsion-free nilpotent group of class 2. Also note that this step is an immediate corollary of our proposition.

PROPOSITION 4.6. *Let G be e.c. in $\mathfrak{N}_{n,\pi}$ and $g \in Z_2$. There exists an element $h \in Z_2$ such that $h^p = g$ if and only if*

(a) *$n = 2$ and g commutes with every element of order p or*

(b) *$n \geqq 3$ and g commutes with every element of order p modulo G_2.*

PROOF. By Lemma 4.3 we have the existence of such an h if and only if g commutes with every element x that satisfies $x^p \in C(Z_2 \cap C(g))$.

If $n \geqq 3$ we have $g \in Z_2 \leqq G_2$ and $C(g) \geqq C(G_2) \geqq Z_2$. Hence $x^p \in C(Z_2 \cap C(g))$ is equivalent to $x^p \in C(Z_2) = G_2$ by Lemma 4.4.

In the case $n = 2$ we have $Z_2 = G$ and again by Lemma 4.4 the range of x^p becomes $C^2(g) = I_{\pi'}(\langle g \rangle)Z$. We assume that $[g,x] \neq 1$ for some x with $x^p \in I_{\pi'}(\langle g \rangle)Z$ and show that there exists some element x'' of order p with $[g,x''] \neq 1$. From the hypothesis it follows that $x^{pq} \in \langle g \rangle Z$ for some π'-number q and hence $x^{pq} = g^k z$ with some $k \in \mathbf{Z}$ and $z \in Z$. Since extraction of π'-roots is unique in the π'-torsion-free group $G \in \mathfrak{N}_{2,\pi}$ and $[g, x^q] = 1$ would imply $[g,x]^q = 1$ and $[g,x] = 1$, we can replace x by x^q and assume $q = 1$. Now $[g,x]^p = [g,x^p] = [g, g^k z] = 1$ and $[g,x]^k = [g^k, x] = [x^p z^{-1}, x] = 1$. Since $[g,x] \neq 1$ we have $p \mid k$, so $k = pk'$. If we take $x' = xg^{-k'}$, then $[g,x'] = [g, xg^{-k'}] = [g,x] \neq 1$ and

$$x'^p = x^p g^{-pk'} [g^{-k'}, x]^{\binom{p}{2}} = z \cdot [g^{-k'}, x]^{\binom{p}{2}} \in Z.$$

Finally we set $x'' = x'z''$ with z'' a p-th root of x'^{-p} in the divisible center Z. Then $[g,x''] = [g,x'] \neq 1$ and $x''^p = x'^p z''^p = 1$. □

We are now ready to investigate the periodic elements of infinite height in an e.c. group in $\mathfrak{N}_{n,\pi}$.

PROPOSITION 4.7. *Let G be e.c. in $\mathfrak{N}_{n,\pi}$ and T its torsion subgroup. The set I of periodic elements of infinite height in G is a divisible subgroup of the center $Z(T)$ of T and allows a decomposition $I = T_n \times T^\infty$, where T^∞ denotes a divisible subgroup of periodic elements which are non-central in G.*

In the case $n = 2$ we have $Z(T) = I = T_n \times T^\infty$.

PROOF. First recall that periodic elements of infinite height lie in the center of T, so $I \subset Z(T)$. Now by Corollary 4.2 we can embed G into some $H \in \mathfrak{N}_{n,\pi}$ such that the periodic elements of infinite height in H form a divisible subgroup of H which clearly contains I. Hence I is a subgroup, since G is e.c., and every element of the p-socle of I has infinite height in H and thus in G, again by the existential closure of G. Therefore I is divisible.

Note that $Z \cap T = G_n \cap T$ is divisible and equal to T_n by fact I. So T_n is contained in I, and we obtain a decomposition $I = T_n \times T^\infty$ where T^∞ is a divisible periodic subgroup and consists of non-central elements, since $Z \cap T = T_n$.

If $n = 2$ it follows from the previous proposition that $Z(T)$ is divisible and hence $Z(T) = I$. □

In the case $n = 2$ we now give a full characterisation of T. Saracino and Wood [16] proved that T is f.g. if and only if $Z(T) = T_2$, which is our case $T^\infty = 1$.

THEOREM 4. *Let G be e.c. in $\mathfrak{N}_{2,\pi}$. The torsion subgroup T has a decomposition as a direct product $T = E_\pi \times T^\infty$ of a f.g. group E_π in $\mathfrak{N}_{2,\pi}$ — this is the same as a periodic e.c. group in $\mathfrak{N}_{2,\pi}$ — and a divisible subgroup T^∞ of non-central elements with respect to G.*

If G is countable then E_π is the direct product of the unique countable f.g. groups in $\mathfrak{N}_{2,p}$ for $p \in \pi$ and thus is itself unique up to isomorphism. The ranks of the p-components of T^∞ are the only invariants of the torsion subgroup in a countable e.c. group in $\mathfrak{N}_{2,\pi}$.

PROOF. As the torsion subgroup T is the direct product of its maximal p-subgroups for $p \in \pi$ and as a direct product of e.c. p-groups in $\mathfrak{N}_{2,p}$ for $p \in \pi$ is e.c. in $\mathfrak{N}_{2,\pi}$ (cf. [7] Satz 5) we may restrict ourselves to the case $\pi = \{p\}$.

Let T^∞ be a divisible subgroup such that $Z(T) = T_2 \times T^\infty$ as in Proposition 4.7. Since $T_2 \cap T^\infty = 1$ and T^∞ is divisible, T^∞ is embedded isomorphically onto a direct factor of T/T_2. If we denote the inverse image of a complement of T^∞ in T/T_2 by E_p, then the first assertion $T = E_\pi \times T^\infty$ follows from $T^\infty \leqq Z(T)$.

Let us now show that E_p is indeed f.g. in $\mathfrak{N}_{2,p}$. Since E_p/T_2 contains no elements of infinite height, every element is contained in a subgroup generated by a finite set $y_1, \cdots, y_m$ of linearly independent elements mod T_2 such that $\langle y_i T_2 \rangle$ is pure in E_p/T_2 $(i = 1, \cdots, m)$. Then $y_1, \cdots, y_m$ generate the direct factor $\langle y_1 T_2 \rangle \times \cdots \times \langle y_m T_2 \rangle$ of E_p/T_2 and moreover of G/G_2. In this situation the main tool of Saracino and Wood [16, lemma 2.1] still works: Given any $z \in T_2$ such that $o(z) \mid o(y_1 T_2)$ one has an automorphism ψ of G sending y_1 to $y_1 z$ and leaving $y_2, \cdots, y_m$ invariant. A split extension of G by such an automorphism ψ then yields the commutator equations $[y_1, \psi] = z$ and $[y_i, \psi] = 1$ $(i = 2, \cdots, m)$ while being still in $\mathfrak{N}_{2,p}$. With this in mind it is also easy to check directly that E_p satisfies the axioms [16, remark 3.10] for an f.g. group in $\mathfrak{N}_{2,p}$. The other assertions are now immediate from the result of Saracino and Wood [16] that, up to isomorphism, there is only one countable f.g. group in $\mathfrak{N}_{2,p}$. □

In the next section we shall prove that the p-rank of T^∞ can assume all cardinal numbers.

5. The number of countable e.c. groups in $\mathfrak{N}_{n,p}$

By means of recursion theoretic methods Hodges [4] showed that for infinite π there exist $2^{\aleph_0}$ non-elementarily equivalent countable e.c. groups in $\mathfrak{N}_{n,\pi}$. Let us embark on this question from a more algebraic point of view and consider the number of non-isomorphic e.c. groups. For any set $X \subset \pi$ Hodges constructed a group $G_X \in \mathfrak{N}_{n,\pi}$ with an element g that is X'-divisible, but not X-divisible mod H_2 for any $H \in \mathfrak{N}_{n,\pi}$ containing G_X. A similar group was studied by B. H. Neumann and Wiegold [12]. Now each G_X may be embedded in a countable e.c. group in $\mathfrak{N}_{n,\pi}$; however, a countable group can contain at most countably many of them. Thus there must exist $2^{\aleph_0}$ non-isomorphic countable e.c. groups in $\mathfrak{N}_{n,\pi}$ for each infinite π.

Let us now deal with the case that π is finite and assume for simplicity $\pi = \{p\}$. Notice that our method will work for any $\pi \neq \emptyset$. Here we start with a divisible group $F \in \mathfrak{N}_{n,p}$ and embed F into an e.c. group $G \in \mathfrak{N}_{n,p}$ of the same cardinality in such a way that the subgroup of G/G_3 which is generated by the divisible elements mod G_2 is isomorphic to F/F_3. This will also answer the question left open after Theorem 4.

Assume $F \in \mathfrak{N}_{n,p}$ and F divisible.

$*(L)$: $F \leqq L \in \mathfrak{N}_{n,p}$, L p'-divisible; for $g \in L \setminus FL_2$ there exists $y \in L$ such that $[g, y] \notin Z_{n-2}(L)$ and $o(y) = p^{1+h(gL_2)}$,

where we denoted by h the p-height, i.e. for $g \in G$ we have $h(g) = \mu$ if g has a p^{μ}-th root in G but not a $p^{\mu+1}$-st one.

The following lemma shows that $*(L)$ prevents elements of L from becoming p-divisible.

LEMMA 5.1. *Let $L \leqq M \in \mathfrak{N}_{n,p}$ with $*(L)$. Then $h(gM_2) = h(gL_2)$ for all $g \in L$. Also a relativized version of $*(L)$ holds:*

$*(L, M)$: *$F \leqq L \leqq M \in \mathfrak{N}_{n,p}$, L p'-divisible; for $g \in L \setminus FM_2$ there exists $y \in L$ such that $[g, y] \notin Z_{n-2}(M)$ and $o(y) = p^{1+h(gL_2)}$.*

PROOF. From $L \cap M_2 \geqq L_2$ we have that $LM_2/M_2 \cong L/(L \cap M_2)$ is a homomorphic image of L/L_2 and $h(gM_2) \geqq h(gL_2)$. Suppose that '$>$' holds for some $g \in L$. As FL_2/L_2 is divisible, it follows that $g \notin L \setminus FL_2$. Hence $*(L)$ gives us $y \in L$ such that $[g, y] \notin Z_{n-2}(L)$ and $o(y) = p^{1+h(gL_2)}$. From $L \cap Z_{n-2}(M) \leqq Z_{n-2}(L)$ we conclude that $[g, y] \notin Z_{n-2}(M)$. By assumption there exist $x \in M$, $c \in M_2$ such that $x^m c = g$ and $p^{1+h(gL_2)} \mid m$. This yields the contradiction modulo $Z_{n-2}(M) \geqq M_3$

$$1 \neq [g, y] = [x^m c, y] = [x^m, y] = [x, y]^m = [x, y^m] = 1.$$

Thus $h(gM_2) = h(gL_2)$ for all $g \in L$. The assertion $*(L, M)$ is now easy. □

The next lemma shows that $*(L)$ can be lifted.

LEMMA 5.2. *Let* $L \leqq M = \langle L, x_1, \cdots, x_k \rangle$ *with* $*(L, M)$. *Let* $* : M \to M/M_2$ *denote the canonical projection.*

(1) *There exist* $a_1, \cdots, a_s \in M$ *such that* $M = \langle L, a_1, \cdots, a_s \rangle$ *and* $M^* = L^* \times \langle a_1^* \rangle \times \cdots \times \langle a_s^* \rangle$.

(2) *There exists a group* $N \geqq M$ *with* $*(N)$.

(3) *If* $*(L_\alpha)$ *and* $L_\alpha \leqq L_\beta$ *hold for* $\alpha < \beta < \kappa$, *then* $*(\bigcup_{\alpha<\kappa} L_\alpha)$ *holds.*

PROOF. (1) Choose $r \leqq s \leqq k$ and $a_1, \cdots, a_s \in M$ such that

$$M = \langle L, a_1, \cdots, a_s \rangle, \langle a_1, \cdots, a_s \rangle^* = \langle a_1^* \rangle \times \cdots \times \langle a_s^* \rangle,$$

$o(a_1^*) \leqq \cdots \leqq o(a_r^*) < \infty$ minimal and $o(a_{r+1}^*) = \cdots = o(a_s^*) = \infty$, $s - r$ minimal.

We first prove that $\langle L, a_1, \cdots, a_r \rangle^* \cap \langle a_{r+1}, \cdots, a_s \rangle^* = 1$. So let $a \in \langle a_{r+1}, \cdots, a_s \rangle$ and $a^{*m} \in \langle L, a_1, \cdots, a_r \rangle^*$. Since the a_i^* are periodic for $i \leqq r$ we may assume $a^{*m} \in L^*$.

If $a^{*m} \in F^*$ we may further assume that a^{*m} lies in the torsion-free direct factor of the divisible abelian group F^*. Then there exists $f \in F$ such that $f^{*-m} = a^{*m}$, or $(af)^{*m} = 1$. As the torsion-free rank of $\langle a_{r+1}, \cdots, a_s \rangle^* / \langle a \rangle^*$ is less than $s - r$, if $a^* \neq 1$, we obtain from the minimality of $s - r$ that $a^* = 1$.

Thus we are left with the case $a^{*m} \in L^* \setminus F^*$. There exists $c \in M_2$ such that $a^m c \in L \setminus FM_2$. From $*(L, M)$ we obtain an element $y \in L$ with $[a^m c, y] \notin Z_{n-2}(M)$ and $o(y) = p^{1+h(a^m cL_2)}$. If $m = p^u t$ with $(p, t) = 1$, it follows from $[a, y^m] \notin Z_{n-2}(M)$ that $o(y) \nmid p^u$, or $u \leqq h(a^m cL_2)$. Since L is p'-divisible there exists $x \in L$ with $x^{-m} = a^m c \bmod L_2$ and hence $(ax)^{*m} = 1$. As above this implies $a^* = 1$, now contradicting $a^{*m} \notin F^*$. Thus we have proved

$$M^* = \langle L, a_1, \cdots, a_r \rangle^* \times \langle a_{r+1} \rangle \times \cdots \times \langle a_s^* \rangle.$$

Let us consider $\langle L, a_1, \cdots, a_r \rangle^*$ and prove that $\langle L, a_2, \cdots, a_r \rangle^* \cap \langle a_1 \rangle^* = 1$. Assume $a_1^{*m} \in \{L, a_2, \cdots, a_r \rangle^*$. Since $o(a_1^*) \leqq o(a_i^*)$, $1 \leqq i \leqq r$, there is an element $a \in \langle a_2, \cdots, a_r \rangle$ such that $(a_1 a)^{*m} \in L^*$. As above we obtain in this situation an element $x \in L$ satisfying $(a_1 a x)^{*m} = 1$. The minimality of $o(a_1^*)$ now yields together with $\langle L, a_1 a x, \cdots, a_r \rangle = \langle L, a_1, \cdots, a_r \rangle$ that $a_1^{*m} = 1$. Inductively we now obtain

$$\langle L, a_1, \cdots, a_r \rangle^* = L^* \times \langle a_1^* \rangle \times \cdots \times \langle a_r^* \rangle$$

and hence the first assertion holds true.

(2) We proceed by induction on s and construct a group $N \geqq M$ such that $\langle L, a_1\rangle \leqq Q \leqq N = \langle Q, a_2, \cdots, a_s\rangle \in \mathfrak{N}_{n,p}$ and $*(Q, N)$ hold. This yields simultaneously the case $s = 1$ and the induction step $s - 1 \rightarrow s$.

For the construction we need finite p-groups $K = K(m) \in \mathfrak{N}_{n,p}$ with an element $c_m \in K_2$, $o(c_m) = o(c_m K_3) = p^m$, and $K_\mu = Z_{n+1-\mu}(K)$ for $1 \leqq \mu \leqq n$. The group K can be chosen, for example, as a group of upper triangular matrices over $\mathbf{Z}/p^m\mathbf{Z}$. Taking $K' \cong K$, we then also see easily that $*(K \times K', M \times K \times K')$ holds (cf. below). Further, let us distinguish an element $f^0 \in K'$ with $o(f^0) = o(f^0 K_2') = p^m$.

We first deal with the case that a_1^* is periodic.

If $o(a_1^*) = p^m$, there exist by Folgerung p. 2183 in [7] $h \in \mathcal{Y} \in \mathfrak{N}_{n,p}$ such that $M \times K \times K' \leqq \mathcal{Y}$, $[a_1, h] = c_m \in K$, $[L, h] = 1$, $o(h) = o(c_m K_3) = p^m$. As the torsion group of $\mathcal{Y}$ is a p-group, we may assume that $\mathcal{Y}$ is p'-divisible. We take $P = \langle L, a_1, K \times K', hf^0\rangle \leqq \mathcal{Y}$, Q the p'-divisible closure of P in $\mathcal{Y}$, and $N = \langle Q, a_2, \cdots, a_s\rangle \leqq \mathcal{Y}$. It is then clear that $M \leqq N$ and Q is p'-divisible. We now check the last clause of $*(Q, N)$.

Let $g' \in Q \setminus FN_2$ and assume first that $g' \in P \setminus FN_2$. Then $g' = gafh' \bmod N_2$ with some $g \in L$, $a \in \langle a_1\rangle$, $f \in K \times K'$, $h' \in \langle hf^0\rangle$, and we may suppose that the decomposition is chosen with a of maximal height $\bmod Q_2$. It follows that $l = h(g'Q_2)$ is the minimum of the respective heights of $g, a, f, h' \bmod Q_2$, because

$$Q/Q_2 \cong \langle L, a_1\rangle Q_2/Q_2 \times (K \times K')Q_2/Q_2 \times \langle hf^0\rangle Q_2/Q_2.$$

If $l = h(gQ_2) < h(aQ_2)$ we obtain $y \in L$ with $[g, y] \notin Z_{n-2}(M)$ and $o(y) = p^{1+h(gL_2)} = p^{1+l}$ from $*(L, M)$. As $h(aQ_2) > l$, it follows that $[a, y] \in Q_3$ and hence $[g', y] = [g, y] \notin Z_{n-2}(N)$. The cases $l = h(fQ_2)$ and $l = h(h'Q_2)$ can be dealt with similarly with $*(K \times K', M \times K \times K')$. If $l = h(aQ_2)$, then $a = a_1^u \bmod Q_2$ for $u = p^l$. Set $v = o(h)/p^{1+l}$. As $o([a_1, h]) = o(h)$, we conclude that $o([a, h^v]) = o([a_1, h]^{uv}) = p \bmod Q_3$ and $[a, h^v] \notin Z_{n-2}(K)$. Since $[a, K'] = 1$ and $[h, K \times K'] \cap K_2 = 1$, we obtain $[a, (hf^0)^v] \notin Z_{n-2}(N)$ and $o((hf^0)^v) = p^{1+l}$.

If now $g' \in Q \setminus FN_2$, then $g'^m \in P \setminus FN_2$ for some m prime to p. We choose $y \in L$ for g'^m with $[g'^m, y] \notin Z_{n-2}(N)$ and $o(y) = p^{1+h(g'^m Q_2)}$. Then also $[g', y] \notin Z_{n-2}(N)$, since $Z_{n-2}(N)$ is p'-isolated, and the p-heights of g' and g'^m are the same $\bmod Q_2$.

Finally consider the case that a_1^* is torsion-free. We then choose $\mathcal{Y}$ as above, but with $h_m \in \mathcal{Y}$ for all $m \geqq 1$, such that

$$[a_1, h_m] = c_m, \qquad \mathcal{Y} \geqq M \times \prod_m (K(m) \times K(m)')$$

and proceed analogously with $P = \langle L, a_1, K(m) \times K(m)', h_m f_m^0;\ m \geqq 1\rangle$.

(3) This is easy by Lemma 5.1. □

Let us now digress briefly to the question of the last section.

PROPOSITION 5.3. *Let κ be a cardinal number. There exists an e.c. group E in $\mathfrak{N}_{2,p}$ such that the rank of T^∞ is κ.*

PROOF. We start with a group from [12]. Let $A \cong B \cong \mathbf{Z}(p^\infty)$ and $\phi : A \to B$ an isomorphism. Then $x : \{a \mapsto aa^\phi,\ b \mapsto b\}$ defines an automorphism of $A \times B$ and the split extension T_1 of $A \times B$ by $\langle x \rangle$ lies in $\mathfrak{N}_{2,p}$. Further $*(T_1)$ holds for $F = A$ and similarly $*(T_\kappa)$ for A_κ, where T_κ and A_κ denote the restricted direct products of κ copies of T_1 and A respectively. Using the previous lemmas T_κ can be embedded by the standard procedure ([3] proposition I.1.3) into an e.c. group $E \in \mathfrak{N}_{2,p}$ with $*(E)$. Thus the maximal divisible subgroup of E/E_2 is $T_\kappa E_2/E_2 \cong A_\kappa$ and has rank κ. □

The same argument yields our final theorem, since there exist 2^κ divisible torsion-free groups $F \in \mathfrak{N}_{n,p}$ of cardinality $\kappa \geqq \aleph_0$ such that the factor groups F/F_3 are non-isomorphic.

THEOREM 5. *In $\mathfrak{N}_{n,p}$ there exist 2^κ e.c. groups of cardinality $\kappa \geqq \aleph_0$. More precisely, for every divisible torsion-free $F \in \mathfrak{N}_{n,p}$ of cardinality κ with $F_\mu = Z_{n+1-\mu}(F)$, $1 \leqq \mu \leqq n$ there exists an e.c. group $E \in \mathfrak{N}_{n,p}$ of cardinality κ such that F/F_3 is isomorphic to the subgroup of E/E_3 generated by those elements of $E \setminus E_2$ which are divisible* mod E_2.

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